State Estimation for Linear Systems with Quadratic Outputs

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The class of systems considered is modeled as

$$\boxed{\begin{aligned} \dot{x} =& Ax + Bu, \\ y_h =& \frac{1}{2} x^\top C_h x + d_h^\top x, \;\; h = 1, \ldots, q \end{aligned}}$$

Without loss of generality, the matrices C_h are assumed to be symmetric.

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Its observability properties are affected by the input u

Example (Observability depends on input)

Impossible to distinguish between initial conditions x_0 and $-x_0$ for

$$\begin{cases} \dot{x} = 0 \\ y = x^2 \end{cases}$$

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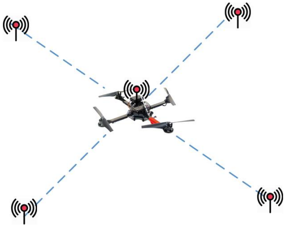
Without loss of generality, the matrices C_h are assumed to be symmetric.

Robot localization from range measurements[1]

$$\begin{cases}\dot{x} = u\\ y_h = \|x - a_h\|^2 - \|a_h\|^2\end{cases}$$



[1] T. Hamel and C. Samson, "Position estimation from direction or range measurements," *Automatica*, vol. 82, pp. 137 –144, 2017.

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Without loss of generality, the matrices C_h are assumed to be symmetric.

We propose a systematic approach to **immerse** the original system into the higher order system

$$\begin{aligned} \dot{z} =& \mathcal{A}(u) x + \mathcal{B} u, \\ y =& \mathcal{C} z \end{aligned}$$

- Incorporating a **minimum number** of auxiliary states
- **Uniform observability** of the pair $(\mathcal{A}(u(t)), \mathcal{C})$ guarantees convergence of Kalman-type filters

Outline

Motivation

Single-Output Systems

Multiple-Output Systems

Conclusion and Future Work

Single-Output Systems

Definition

- $\mathbb{M}_n \subset \mathbb{R}^{n \times n}$ denote the space of real $n \times n$ symmetric matrices
- The Lyapunov operator $\mathbf{L}_A : \mathbb{M}_n \to \mathbb{M}_n$ is defined by

$$\mathbf{L}_A(X) := XA + A^\top X$$

Single-Output Systems

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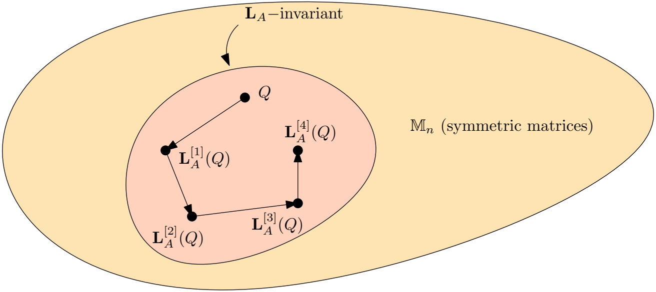
Observation: Any *quadratic form*, along the trajectories of the system, satisfies

$$\frac{1}{2}\frac{d}{dt}(x^\top Qx) = \mathbf{\frac{1}{2}x^\top L_A(Q)x} + (QBu)^\top x, \tag{1}$$

for some symmetric matrix $Q \in \mathbb{M}_n$.

Single-Output Systems



Lemma

For any $Q \in \mathbb{M}_n$, **there exists a minimum number** *m such that $m \leq n(n+1)/2$ and*

$$\mathbb{Q} := \mathrm{span}\{\mathbf{L}_A^{[0]}(Q), \mathbf{L}_A^{[1]}(Q), \dots, \mathbf{L}_A^{[m-1]}(Q)\}$$

is $\mathbf{L}_A$−invariant.

Single-Output Systems

Example

Consider the double-integrator system on $\mathbb{R}^n$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = 0.5\|x_1\|^2 \end{cases}$$

We have

$$\mathbf{L}_A^{[0]}(C) = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{L}_A^{[1]}(C) = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad \mathbf{L}_A^{[2]}(C) = \begin{bmatrix} 0 & 0 \\ 0 & 2I_n \end{bmatrix}$$

and finally, $\mathbf{L}_A^{[3]}(C) = 0$. *Hence*

$$\mathrm{span}\{\mathbf{L}_A^{[0]}(Q), \mathbf{L}_A^{[1]}(Q), \mathbf{L}_A^{[2]}(Q)\}$$

is $\mathbf{L}_A-$*invariant.*

Single-Output Systems

Example

Consider the double-integrator system on $\mathbb{R}^n$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 + 2x_2 + u \\ y = 0.5x_1^2 + 0.5x_2^2 \end{cases}$$

We have

$$\mathbf{L}_A^{[1]}(C) = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{L}_A^{[2]}(C) = \begin{bmatrix} 4 & 8 \\ 8 & 20 \end{bmatrix}$$

and finally, $\mathbf{L}_A^{[2]}(C) = 4(\mathbf{L}_A^{[0]}(C) + \mathbf{L}_A^{[1]}(C))$. *Hence*

$$\mathrm{span}\{\mathbf{L}_A^{[0]}(Q), \mathbf{L}_A^{[1]}(Q)\}$$

is $\mathbf{L}_A$*−invariant. In this case, the number of auxiliary states needed is* $m = 2$*, which is strictly less than* $n(n+1)/2 = 3$.

Single-Output Systems

Let us define the following **auxiliary state variables**

$$\boxed{\xi_k := \frac{1}{2} x^\top \mathbf{L}_A^{[k]}(C) x}, \quad k = 0, 1, \cdots, (m-1), \tag{2}$$

[2] By the previous lemma, there exist α_k s.th. $\mathbf{L}_A^{[m]}(Q) = \sum_{k=0}^{m-1} \alpha_k \mathbf{L}_A^{[k]}(Q)$

Single-Output Systems

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One has[2]

$$\begin{aligned} \dot{\xi}_k &= \xi_{k+1} + (Bu)^\top \mathbf{L}_A^{[k]}(C)x, \;\; k = 0, \ldots, (m-2) \\ \dot{\xi}_{m-1} &= \sum_{k=0}^{m-1} \alpha_k \xi_k + (Bu)^\top \mathbf{L}_A^{[m-1]}(C)x. \end{aligned} \tag{3}$$

[2] By the previous lemma, there exist α_k s.th. $\mathbf{L}_A^{[m]}(Q) = \sum_{k=0}^{m-1} \alpha_k \mathbf{L}_A^{[k]}(Q)$

Single-Output Systems

Define $z := \begin{bmatrix} \xi_{m-1} & \cdots & \xi_1 & \xi_0 & x^\top \end{bmatrix}^\top \in \mathbb{R}^{m+n}$:

$$\boxed{\begin{aligned} \dot{z} =& \mathcal{A}(u)x + \mathcal{B}u, \\ y =& \mathcal{C}z \end{aligned}}$$

$$\mathcal{A}(u) = \left[\begin{array}{ccccc|c} \alpha_{m-1} & \alpha_{m-2} & \dots & \alpha_1 & \alpha_0 & (Bu)^\top \mathbf{L}_A^{[m-1]}(C) \\ 1 & 0 & \dots & 0 & 0 & (Bu)^\top \mathbf{L}_A^{[m-2]}(C) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & (Bu)^\top \mathbf{L}_A^{[0]}(C) \\ \hline 0 & 0 & \cdots & 0 & 0 & A \end{array}\right],$$

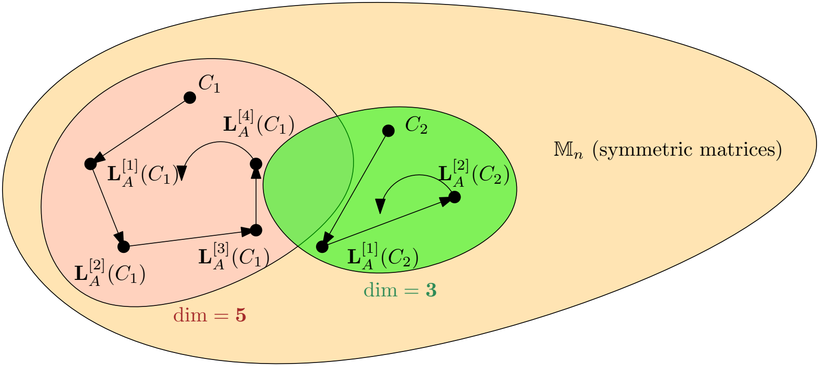
$$\mathcal{B} := \left[\begin{array}{ccc|c} 0 & 0 & \cdots & B^\top \end{array}\right]^\top, \quad \mathcal{C} := \left[\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & 1 & d^\top \end{array}\right]$$

<u>Uniform observability has been</u> discussed in[3] when $\alpha_k = 0$

[3] D. Theodosis, S. Berkane, and D. V. Dimarogonas, "State estimation for a class of linear systems with quadratic output," *IFAC-PapersOnLine*, vol. 54, no. 9, pp. 261–266, 2021.

Multiple-Output Systems

The naive extension of the single-output procedure to multiple-outputs does not guarantee the minimum number of auxiliary states



$$\begin{aligned}\dim(\mathbb{C}_1 + \mathbb{C}_2) &= \dim(\mathbb{C}_1) + \dim(\mathbb{C}_2) - \dim(\mathbb{C}_1 \cap \mathbb{C}_2) \\ &\leq \dim(\mathbb{C}_1) + \dim(\mathbb{C}_2)\end{aligned}$$

Multiple-Output Systems [4]

$$y_h = \frac{1}{2} x^\top C_h x + d_h^\top x \tag{4}$$

4

Multiple-Output Systems [4]

$$
\begin{aligned}
y_h &= \frac{1}{2} x^\top C_h x + d_h^\top x \\
&= \frac{1}{2} \mathrm{vec}(C_h)^\top (x \otimes x) + d_h^\top x
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&= \frac{1}{2}\mathrm{vech}(C_h)^\top D_n^\top (x \otimes x) + d_h^\top x.
\end{aligned} \tag{4}
$$

[4] For $A \in \mathbb{M}_n$, $\mathrm{vech}(A)$ is obtained by vectorizing only the lower triangular part of A. One has $D_n \mathrm{vech}(A) = \mathrm{vec}(A)$ where D_n is the *duplication matrix*.

Multiple-Output Systems [4]

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$$

Example

$$
(x \otimes x) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2 x_1 \\ x_2^2 \end{bmatrix}, \quad D_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow x^{[2]} = \begin{bmatrix} x_1^2 \\ 2x_1 x_2 \\ x_2^2 \end{bmatrix}
$$

[4] For $A \in \mathbb{M}_n$, $\mathrm{vech}(A)$ is obtained by vectorizing only the lower triangular part of A. One has $D_n \mathrm{vech}(A) = \mathrm{vec}(A)$ where D_n is the *duplication matrix*

Multiple-Output Systems

The output vector $y := [y_1 \cdots y_q]^\top$ can be written as

$$y = \frac{1}{2} \begin{bmatrix} \mathrm{vech}(C_1)^\top \\ \vdots \\ \mathrm{vech}(C_q)^\top \end{bmatrix} x^{[2]} + \begin{bmatrix} d_1^\top \\ \vdots \\ d_q^\top \end{bmatrix} x =: \bar{C} x^{[2]} + Dx. \tag{5}$$

Lemma

Along the trajectories of the system, one has

$$\frac{d}{dt} x^{[2]} = \bar{A} x^{[2]} + \bar{U}(t) x. \tag{6}$$

with $\bar{A} := D_n^\top (A \oplus A)(D_n^+)^\top$ *and* $\bar{U}(t) := D_n^\top (Bu(t) \oplus Bu(t))$.

- The extended state $z^\top := [(x^{[2]})^\top \; x^\top]$ leads to an LTV system. The number of added auxiliary states is $n(n+1)/2$.

Multiple-Output Systems

- In this work, we propose a **procedure to add the minimum number of auxiliary states** to form an LTV system.
- Rank factorization[5]

$$\frac{1}{2}\begin{bmatrix}\mathrm{vech}(C_1)^\top \\ \vdots \\ \mathrm{vech}(C_q)^\top\end{bmatrix} = \bar{C} = F\bar{L}_0, \quad \mathrm{rank}(\bar{L}_0) = \mathrm{rank}(\bar{C}) = p_0$$

- Let $\xi_0 \in \mathbb{R}^{p_0}$ such that

$$\xi_0 := \bar{L}_0 x^{[2]} = \begin{bmatrix}\frac{1}{2}x^\top L_1 x & \cdots & \frac{1}{2}x^\top L_{p_0} x\end{bmatrix}^\top . \qquad (7)$$

- We have[6]

$$\dot{\xi}_0 = \bar{L}_0 \bar{A} x^{[2]} + \bar{L}_0 \bar{U}(t) x. \qquad (8)$$

[5] $\bar{L}_0$ is full row rank and F is full column rank.
[6] $\dot{\xi}_0$ will be linear in ξ_0 and x if and only if $\mathrm{span}\{L_1, \cdots, L_{p_0}\}$ is $\mathbf{L}_A$−invariant

Multiple-Output Systems

- Assume

$$p_1 := \mathrm{rank}\begin{bmatrix}\bar{L}_0 \\ \bar{L}_0\bar{A}\end{bmatrix} - p_0 \neq 0 \tag{9}$$

- There exist $P_0, M_0^{(0)}, \bar{L}_1$ such that[7]

$$P_0\bar{L}_0\bar{A} = \begin{bmatrix}\bar{L}_1 \\ M_0^{(0)}P_0\bar{L}_0\end{bmatrix}. \tag{10}$$

- If we let $\xi_1 := \bar{L}_1 x^{[2]}$, it follows that

$$P_0\dot{\xi}_0 = \begin{bmatrix}0 \\ M_0^{(0)}\end{bmatrix} P_0\xi_0 + \begin{bmatrix}I_{p_1} \\ 0\end{bmatrix}\xi_1 + P_0\bar{L}_0\bar{U}x. \tag{11}$$

[7] P_k is a permutation matrix

Multiple-Output Systems

Algorithm 1 Computation of the matrices $\bar{L}_k, P_k, M_i^{(k)}$

Require: Output matrix $\bar{C}$ and matrix $\bar{A}$

Compute $p_0 := \text{rank}(\bar{C})$
Rank factorize $\bar{C} = F\bar{L}_0$ with $\text{rank}(\bar{L}_0) = p_0$
for $k \in \{1, 2, \cdots\}$ **do**
  Calculate

$$p_k := \text{rank} \begin{bmatrix} \bar{L}_0 \\ \vdots \\ \bar{L}_{k-1} \\ \bar{L}_{k-1}\bar{A} \end{bmatrix} - \sum_{i=0}^{k-1} p_i \qquad (30)$$

  Find a permutation matrix $P_{k-1} \in \mathbb{R}^{p_{k-1} \times p_{k-1}}$, matrices $M_i^{(k-1)} \in \mathbb{R}^{(p_{k-1}-p_k) \times p_i}$ and a full row matrix $\bar{L}_k \in \mathbb{R}^{p_k \times n(n+1)/2}$ satisfying[1]

$$P_{k-1}\bar{L}_{k-1}\bar{A} = \begin{bmatrix} \bar{L}_k \\ \sum_{i=0}^{k-1} M_i^{(k-1)} P_i \bar{L}_i \end{bmatrix} \qquad (31)$$

  if $p_k = 0$ **then**
    Define $m = k$
    return the matrices $\bar{L}_{k-1}, P_{k-1}, M_{i-1}^{(k-1)}$ for all $1 \leq i \leq k \leq m$.
  end if
end for

Multiple-Output Systems

We define the *extended state vector*

$$z := \begin{bmatrix} P_{m-1}\xi_{m-1} \\ \vdots \\ P_0\xi_0 \\ x \end{bmatrix} := \begin{bmatrix} P_{m-1}\bar{L}_{m-1}x^{[2]} \\ \vdots \\ P_0\bar{L}_0x^{[2]} \\ x \end{bmatrix} \in \mathbb{R}^{n+\sum_{k=0}^{m-1} p_k}, \qquad (12)$$

the dynamics of z are an LTV system with matrices

$$\mathcal{A}(u) = \left[\begin{array}{cccc|c} \bar{M}_{m-1}^{(m-1)} & \cdots & \cdots & \bar{M}_0^{(m-1)} & P_{m-1}\bar{L}_{m-1}\bar{U} \\ \bar{P}_{m-1} & \cdots & \cdots & \bar{M}_0^{(m-2)} & P_{m-2}\bar{L}_{m-2}\bar{U} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \bar{P}_1 & M_0^{(0)} & P_0\bar{L}_0\bar{U} \\ \hline & & 0 & & A \end{array}\right], \qquad (13)$$

$$\mathcal{B} = \left[\begin{array}{c} 0 \\ \hline B \end{array}\right], \ \mathcal{C} = \left[\begin{array}{cccc|c} 0 & \cdots & 0 & FP_0^\top & D \end{array}\right], \qquad (14)$$

Multiple-Output Systems

Example

Consider the system

$$\begin{cases} \dot{p} &= v_a + v_w \\ \dot{v}_a &= u(t) \\ \dot{v}_w &= 0 \end{cases} \qquad \begin{cases} y_1 &= 0.5\|p\|^2 \\ y_2 &= 0.5\|v_a\|^2 \end{cases}$$

The proposed algorithm returns $m = 3$ with

$$\xi_2 = 2\|v_a + v_w\|^2$$
$$\xi_1 = 2p^\top(v_a + v_w)$$
$$\xi_0 = \begin{bmatrix} \|p\|^2 \\ \|v_a\|^2 \end{bmatrix}$$

These auxiliary states allow to bring the system dynamics to an LTV form where a Kalman-type estimator can be applied.

Multiple-Output Systems

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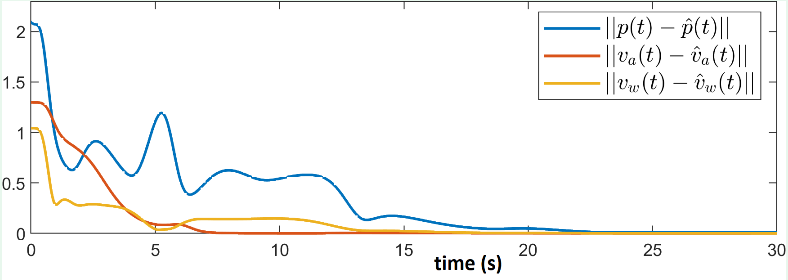

Conclusion and Future Work

Summary:

- An immersion-type technique that transforms LTI systems with quadratic outputs to an LTV system with linear output
- The approach guarantees that only the minimum number of auxiliary states is introduced
- The resultant systems observability is tightly related to the richness of the input signal

Future work:

- Extend the approach to LTV systems with quadratic outputs
- Extend the approach to polynomial outputs
- Design reduced-order observers to estimate only the original state

Thank you

Questions?