Application: Pose Estimation

Workshop 10 – Geometric Observers on Manifolds and Lie-Groups

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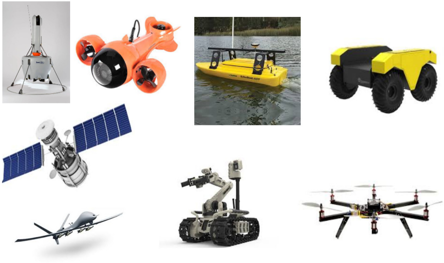
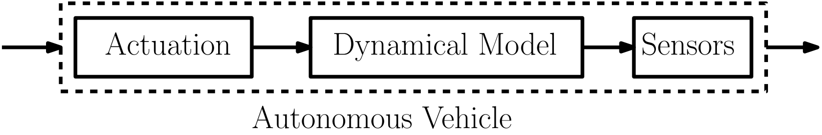
Motivation

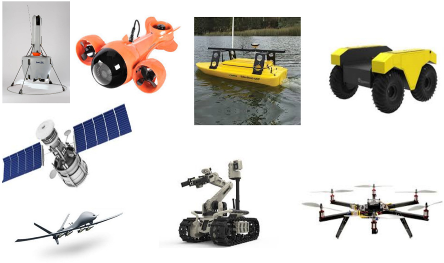
What is common to these vehicles?





Motivation

What is common to these vehicles?

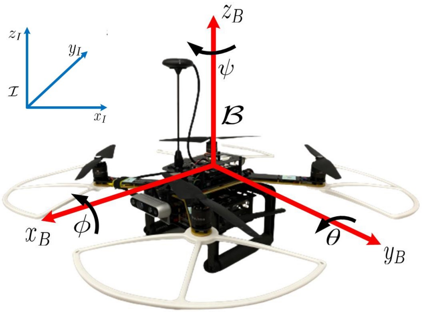


Autonomous Vehicle

- The configuration of any vehicle (rigid body) can be represented by a rotation matrix $R \in \mathbb{SO}(3)$ and a position vector $p \in \mathbb{R}^3$ such that

$$\begin{cases} \dot{R} &= R\omega_\times, \\ \dot{p} &= RV, \end{cases}$$

where ω and V are respectively the angular and linear velocities expressed in body-frame.



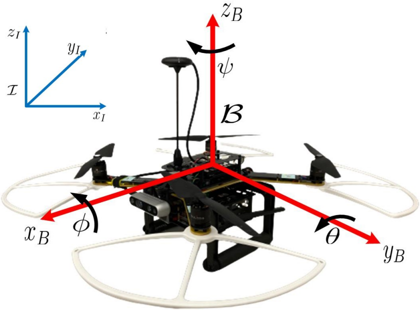
Motivation

Dynamical Model

- The motion of a rigid body in an ideal fluid satisfies the Euler-Lagrange equations

$$\begin{cases} \mathbf{J}\dot{\omega} & = \mathbf{J}\omega_\times\omega + \mathbf{M}V_\times V + f_\omega, \\ \mathbf{M}\dot{V} & = \mathbf{M}V_\times\omega + f_V, \end{cases}$$

where $[f_\omega, f_V]$ is the resultant generalized force acting on the main body.



J and **M** respectively represent the inertia matrix and the added mass matrix.

Outline

Inertial Measurement Unit (IMU)

$$\text{gyroscope:} \quad \omega^y = \omega, \tag{1}$$

$$\text{accelerometer:} \quad a^{\mathcal{B}} = R^\top(\dot{v} - g), \tag{2}$$

$$\text{magnetometer (Optional):} \quad m^{\mathcal{B}} = R^\top m, \tag{3}$$

- m: a constant and know earth's magnetic field
- g: inertial-frame acceleration due to gravity
- v: inertial-frame linear velocity

remark

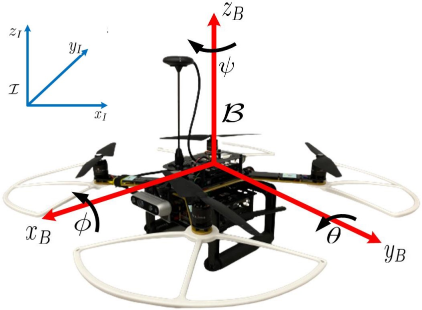
For simplicity, we will consider bias-free measurements in this talk.

Inertial Measurement Unit (IMU)

IMU Kinematics

From the dynamical model of the rigid-body:

$$\begin{cases} \dot{p} &= v, \\ \dot{v} &= g + R a^{\mathcal{B}}, \\ \dot{R} &= R \omega_\times, \end{cases}$$



Inputs are $(a^{\mathcal{B}}, \omega)$

Objective

Estimate the **extended pose** (p, v, R) *using measurements* $y = h(p, R)$.

- The estimators proposed here **evolve on** $SO(3) \times \mathbb{R}^n$

From the dynamical model of the rigid-body:

$$\begin{cases} \dot{p} &= v, \\ \dot{v} &= g + Ra^{\mathcal{B}}, \\ \dot{R} &= R\omega_\times, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \dot{p}^{\mathcal{B}} &= -[\omega]_\times p^{\mathcal{B}} + v^{\mathcal{B}}, \\ \dot{v}^{\mathcal{B}} &= -[\omega]_\times v^{\mathcal{B}} + a^{\mathcal{B}} + g^{\mathcal{B}}, \\ \dot{R} &= R\omega_\times, \end{cases}$$

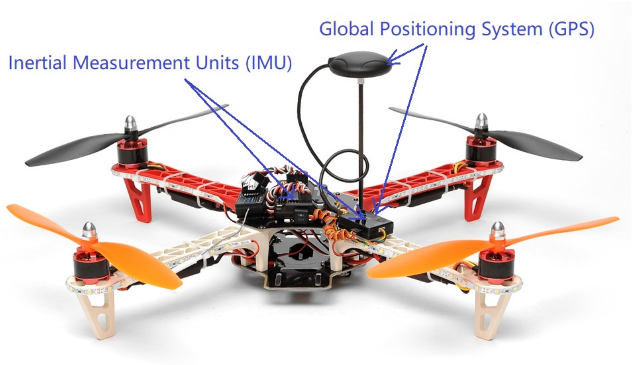
Coupling is through:

- $a := Ra^{\mathcal{B}}$: unknown apparent acceleration, capturing all non-gravitational forces applied to the vehicle, expressed in $\{I\}$.
- $g^{\mathcal{B}} := R^\top g$: unknown body-frame gravitational forces.

Outline

Inertial-Frame Position Information

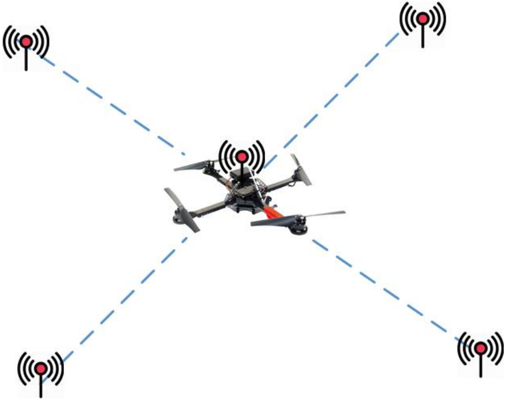
Full position (*e.g.*, GPS)



$$y = p$$

Inertial-Frame Position Information

Range measurements (*e.g.*, Ultra-wideband (UWB))



$$d_i = \|p - p_i\|, \quad i = 1, \cdots, n.$$

Inertial-Frame Position Information

Range measurements (*e.g.*, Ultra-wideband (UWB))

At least 4 non-coplanar source points are needed located at p_i.

$$d_i = \|p - p_i\|, \quad i = 1, \cdots, n.$$

Output equaion:

$$y_i := \frac{1}{2}\left(d_i^2 - d_1^2 - \|p_i\|^2 + \|p_1\|^2\right), \quad i = 2, \cdots, n.$$

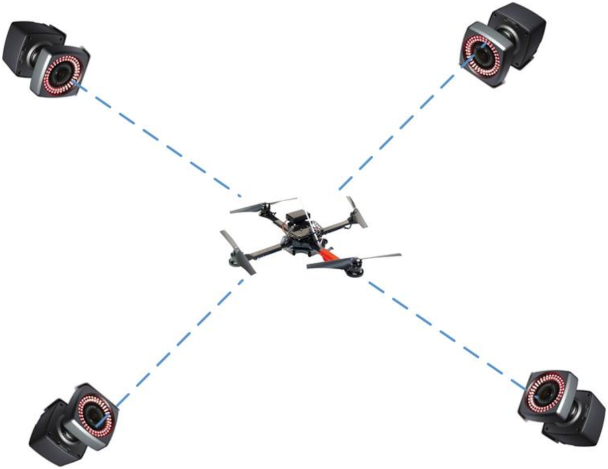
$$y = \begin{bmatrix} (p_1 - p_2)^\top \\ \vdots \\ (p_1 - p_n)^\top \end{bmatrix} p := C_p p \tag{4}$$

Assumption (Observability)

$\mathrm{rank}(C_p) = 3.$

Inertial-Frame Position Information

Bearing measurements (*e.g.*, motion capture system)



Inertial-Frame Position Information

Bearing measurements (*e.g.*, motion capture system)

$$b_i = R_i^\top \frac{p - p_i}{\|p - p_i\|}, \quad i = 1, \cdots, n,$$

- $R_i \in \mathbb{SO}(3)$: Orientation of camera i w.r.t. the inertial frame.
- p_i: Position of camera i in the inertial frame.

Inertial-Frame Position Information

Bearing measurements (*e.g.*, motion capture system)

$$b_i = R_i^\top \frac{p - p_i}{\|p - p_i\|}, \quad i = 1, \cdots, n,$$

- $R_i \in \mathbb{SO}(3)$: Orientation of camera i w.r.t. the inertial frame.
- p_i: Position of camera i in the inertial frame.
- Note that

$$\Pi(b_i) R_i^\top (p - p_i) = 0$$

where $\Pi(z) := I - zz^\top$ is the orthogonal projection.

Inertial-Frame Position Information

Bearing measurements (*e.g.*, motion capture system)

$$b_i = R_i^\top \frac{p - p_i}{\|p - p_i\|}, \quad i = 1, \cdots, n,$$

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- Note that

$$\Pi(b_i) R_i^\top (p - p_i) = 0$$

  where $\Pi(z) := I - zz^\top$ is the orthogonal projection.
- Therefore, we obtain the following output vector

$$y = \begin{bmatrix} \Pi(b_1) R_1^\top p_1 \\ \vdots \\ \Pi(b_n) R_n^\top p_n \end{bmatrix} = \begin{bmatrix} \Pi(b_1) R_1^\top \\ \vdots \\ \Pi(b_n) R_n^\top \end{bmatrix} p := C_p(t) p.$$

Outline

Body-Frame Position Information

Vision: Stereo vs. Monocular

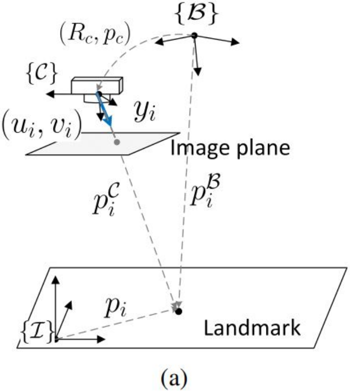


Figure 1: (a) monocular vision system (b) stereo vision system.

Body-Frame Position Information

Vision: Stereo vs. Monocular

Let p_i denote the (constant and known) position of the i-th landmark in frame $\{\mathcal{I}\}$, and

$$p_i^{\mathcal{B}} := R^\top (p_i - p)$$

denote the position of the i-th landmark in frame $\{\mathcal{B}\}$

1) **Stereo-bearing measurements:**

$$y_i^s := \frac{p_i^{\mathcal{C}_s}}{\|p_i^{\mathcal{C}_s}\|} = \frac{R_{cs}^\top (p_i^{\mathcal{B}} - p_{cs})}{\|p_i^{\mathcal{B}} - p_{cs}\|}, \ i \in \{1, 2, \ldots, N\}, s \in \{1, 2\} \tag{5}$$

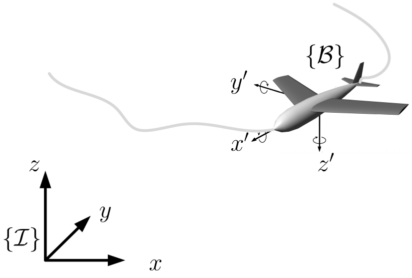
where $p_i^{\mathcal{C}_s} = R_{cs}^\top (p_i^{\mathcal{B}} - p_{cs})$.

2) **Monocular-bearing measurements:**

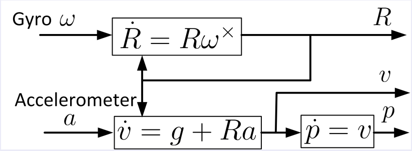
$$y_i := \frac{p_i^{\mathcal{C}}}{\|p_i^{\mathcal{C}}\|} = \frac{R_c^\top (p_i^{\mathcal{B}} - p_c)}{\|p_i^{\mathcal{B}} - p_c\|}, \ i \in \{1, 2, \ldots, N\} \tag{6}$$

where $p_i^{\mathcal{C}} = R_c^\top (p_i^{\mathcal{B}} - p_c)$.

Outline



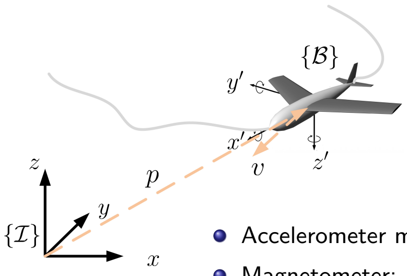
Inertial odometry



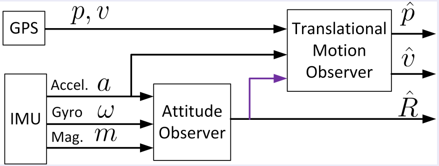
- Determination of the attitude, position and linear velocity
- Relies on the knowledge of the initial conditions
- Sensitive to low frequency noise (drift caused by the integration)

Inertial Navigation Systems

GPS-Aided Inertial Navigation



Cascaded Observer



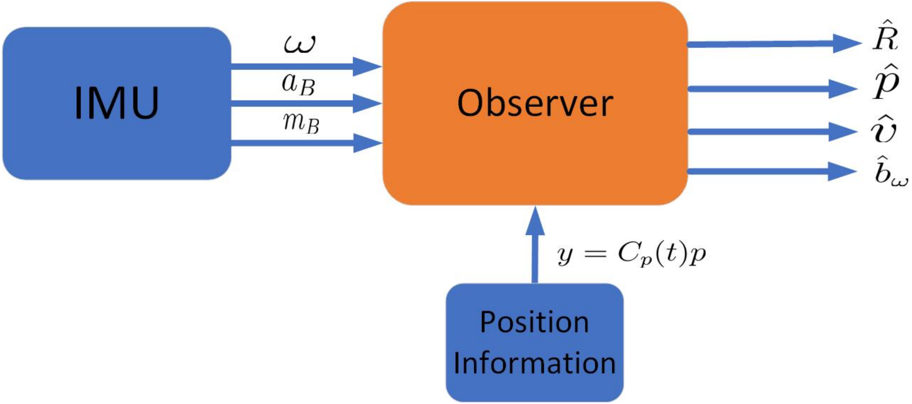
- Accelerometer measurements: $a^{\mathcal{B}} \approx -R^\top g$ (*i.e.*, $\dot{v} \approx 0$)
- Magnetometer: $m^{\mathcal{B}} = R^\top m$
- Independent attitude observer relying only on IMU measurements
- Translational motion observer depends on the estimated attitude $\hat{R}$
- **NOT** reliable during aggressive dynamic maneuvers

T. H. Bryne, J. M. Hansen, R. H. Rogne, *et al.*, "Nonlinear observers for integrated INS/GNSS navigation: Implementation aspects," *IEEE Control Systems Magazine*, vol. 37, no. 3, pp. 59–86, 2017.

S. Bonnabel, P. Martin, and P. Rouchon, "Symmetry-preserving observers," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2514–2526, 2008.

Outline

Universal Observers: Inertial-Frame Position Information



Universal Observers: Inertial-Frame Position Information

Translational Dynamics

Let $x := (p, v) \in \mathbb{R}^6$. The dynamics of x are written as:

$$\dot{x} = Ax + B(g + a), \tag{7}$$

$$y = Cx, \tag{8}$$

where the matrices A, B and C are defined as follows:

$$A = \begin{bmatrix} 0_{3\times3} & I_3 \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, B := \begin{bmatrix} 0_{3\times3} \\ I_3 \end{bmatrix}, C = \begin{bmatrix} C_p^\top \\ 0_{3\times m} \end{bmatrix}^\top. \tag{9}$$

- Linear time-invariant system with unknown input a
- We only measure $a^{\mathcal{B}} = R^\top a$ (in body-frame)

Assumption

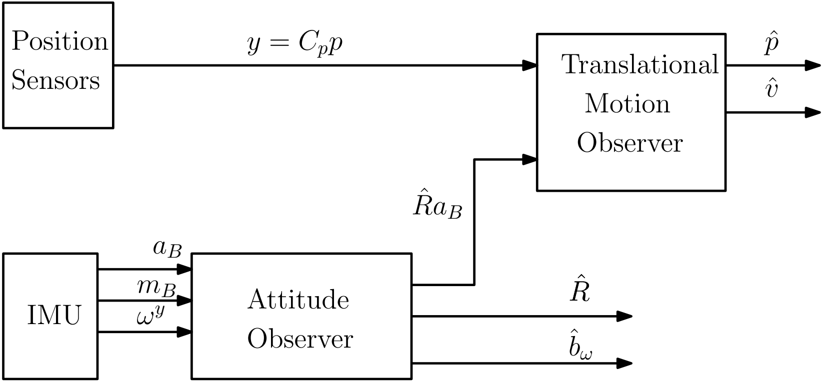
C is constant (e.g., UWB range or GPS position measurements)

Universal Observers: Inertial-Frame Position Information

Common estimation approach (ad-hoc method)

Assumption

Negligible acceleration, i.e., $a \approx -g$.



Assumption

Negligible acceleration, i.e., $a \approx -g$.

$$\text{Attitude observer} \begin{cases} \dot{\hat{R}} &= \hat{R}[\omega + \sigma_R]_\times, \\ \sigma_R &= \rho_1(m_B \times \hat{R}^\top m) + \rho_2(a^{\mathcal{B}} \times \hat{R}^\top(-g)). \end{cases}$$

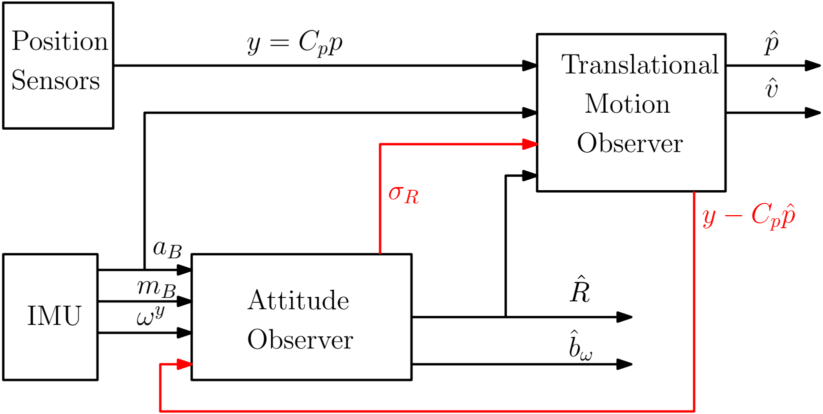
$$\text{Translational Observer} \begin{cases} \dot{\hat{x}} = A\hat{x} + B(g + \hat{R}a^{\mathcal{B}}) + K(y - C\hat{x}) \\ (A - KC) \text{ is Hurwitz} \end{cases}$$

R. Mahony, T. Hamel and J. Pflimlin, "**Nonlinear Complementary Filters on the Special Orthogonal Group**," in IEEE TAC, 2008.

H. F. Grip *et al.* "**Attitude Estimation Using Biased Gyro and Vector Measurements With Time-Varying Reference Vectors**," in IEEE TAC, 2012.

Universal Observers: Inertial-Frame Position Information

Proposed nonlinear observer



- The proposed observer introduces coupling between the translational estimator and the rotational estimator through their innovation terms.
- This additional coupling is important to guarantee the stability of the observer **without the "small" acceleration assumption**.

Universal Observers: Inertial-Frame Position Information

Translational Motion Observer

Let $K = L_\gamma K_0$ with $L_\gamma = \mathrm{blockdiag}(\gamma I_3, \gamma^2 I_3)$ and $\gamma \geq 1$:

$$\dot{\hat{x}} = A\hat{x} + B(g + \hat{R}a^{\mathcal{B}}) + K(y - C\hat{x}) + \sigma_x$$

where $(A - K_0 C)$ is Hurwitz and

$$\sigma_x = -k_R (A - KC)^{-1} B [\hat{R}\sigma_R]_\times \hat{R} a^{\mathcal{B}}$$

- The matrix L_γ is introduced to assign a certain **time-scaling structure** between the different estimation errors.
- The expression of σ_x is obtained through Lyapunov analysis.

Consider the "nonlinear complementary filter"-type observer

$$\dot{\hat{R}} = \hat{R}[\omega + \sigma_R]_\times$$

with

$$\sigma_R = \rho_1(m_B \times \hat{R}^\top m_I) + \rho_2(a^{\mathcal{B}} \times \hat{R}^\top \mathbf{sat}(B^\top K(y - C\hat{x}))).$$

- **sat**: is a saturation function

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H. F. Grip, T. I. Fossen, T. A. Johansen and A. Saberi, "**Attitude Estimation Using Biased Gyro and Vector Measurements With Time-Varying Reference Vectors**," in IEEE TAC, 2012.

- Estimation errors:

$$\tilde{R} := R\hat{R}^\top, \quad \tilde{x} := x - \hat{x}. \tag{10}$$

- **Coupled** auxiliary error variable:

$$\zeta := L_\gamma^{-1}\left[(A - KC)\tilde{x} + B(I - \tilde{R})^\top a\right]. \tag{11}$$

- Attitude innovation term σ_R:

$$\sigma_R = \rho_1(\hat{R}m^{\mathcal{B}} \times m) + \rho_2(\hat{R}a^{\mathcal{B}} \times a) - \rho_2\gamma(\hat{R}a^{\mathcal{B}} \times B^\top\zeta). \tag{12}$$

- Closed-loop dynamics

$$\dot{\tilde{R}} = -\tilde{R}[\sigma_R(\tilde{R}, \zeta, t)]_\times, \tag{13}$$

$$\frac{1}{\gamma}\dot{\zeta} = (A - PC^\top QC)\zeta + \frac{1}{\gamma^2}B(I - \tilde{R})^\top \dot{a}(t). \tag{14}$$

Assumption

- *m and $a(t)$ are not collinear for all times*
- *$\omega, a, \dot{a}$ are uniformly bounded*

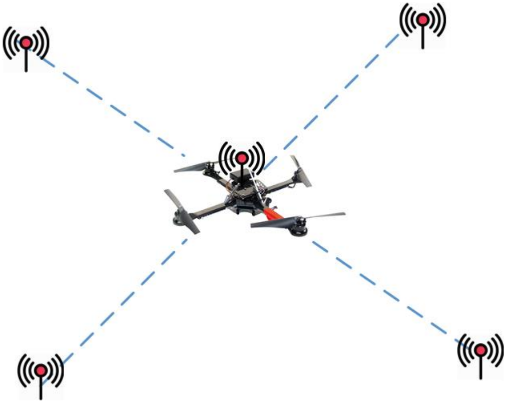
Theorem (Almost semi-global exponential stability)

For all initial conditions (except attitude errors at 180°), there exist (high) gains such that the estimation error converges exponentially to zero.

- Although, sufficiently large gains are needed in the proof, simulation results demonstrate that this is very conservative.
- Without magnetometer, similar result is obtained if $Ra^{\mathcal{B}}$ is PE

Universal Observers: Inertial-Frame Position Information

Simulation Results: Range measurements



- Circular trajectory (with increasing acceleration):

$$p(t) = \begin{bmatrix} \cos(2\pi t^2/100) \\ \sin(2\pi t^2/100) \\ 1 \end{bmatrix}. \tag{15}$$

- Angular velocity:

$$\omega(t) = \begin{bmatrix} \sin(0.2t) \\ \cos(0.1t) \\ \sin(0.3t + \pi/6) \end{bmatrix}, \tag{16}$$

- Anchors:

$$p_1 = [1\ 1\ 2]^\top, \tag{17}$$

$$p_2 = [1\ 3\ \ 0]^\top, \tag{18}$$

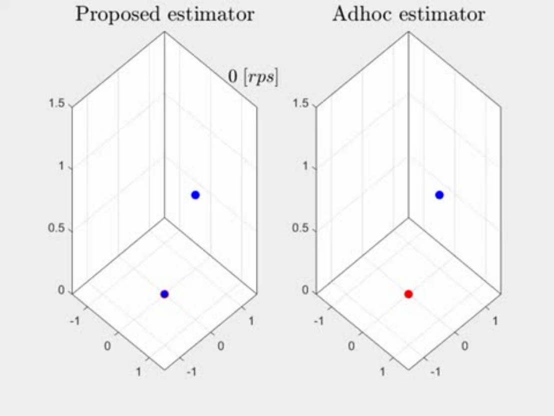
$$p_3 = [0\ 1\ 1]^\top, \tag{19}$$

$$p_4 = [6\ 5\ 5]^\top. \tag{20}$$

Universal Observers: Inertial-Frame Position Information

Simulation Results: Range measurements

Estimated trajectory (red)
True trajectory (blue)



Universal Observers: Inertial-Frame Position Information

Simulation Results: Range measurements

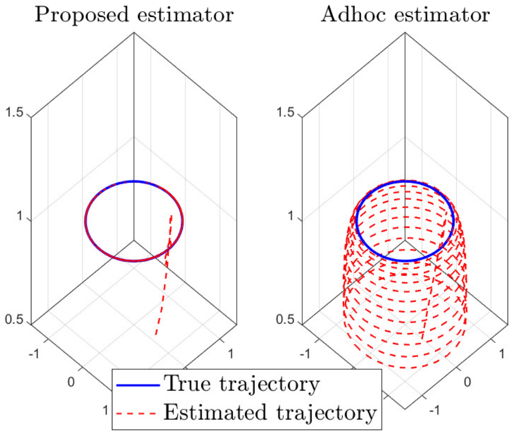


Figure 2: As the acceleration increases, the adhoc estimator drifts away from the true trajectory while the proposed estimator is stable.

Universal Observers: Inertial-Frame Position Information

Simulation Results: Range measurements

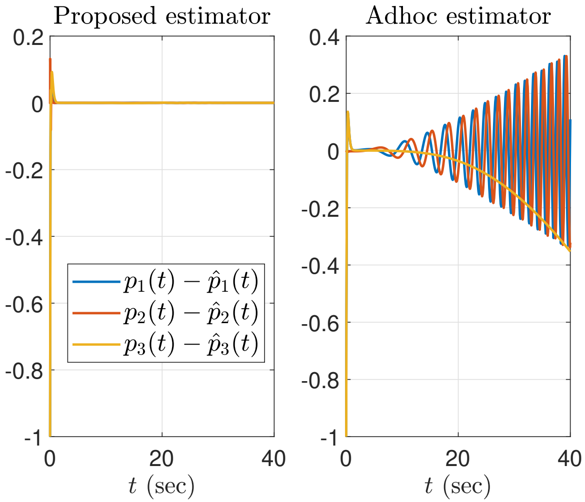


Figure 3: Position estimation errors.

Let $x := (p, v, a) \in \mathbb{R}^9$.

$$\dot{x} = Ax + B_1 g + B_2 \dot{a}$$
$$y = C(t)x$$

$$A = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, B_1 := \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, B_2 := \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad C(t) = \begin{bmatrix} C_p(t) \\ 0 \\ 0 \end{bmatrix}^\top$$

- Linear system with unknown input $\dot{a}$ (jerk)
- We only measure $a^{\mathcal{B}} = R^\top a$

Consider the nonlinear observer

$$\begin{cases} \hat{x} &= \hat{z} + B_2 \hat{R} a^{\mathcal{B}} \\ \dot{\hat{z}} &= A\hat{x} + B_1 g + K(t)(y - C(t)\hat{x}) - B_2 \hat{R} [\sigma_R]_\times a^{\mathcal{B}} \end{cases}$$

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- If $\hat{R} \to R$, this is equivalent to the **Luenberger-type observer**

$$\dot{\hat{x}} = A\hat{x} + B_1 g + B_2 \dot{a} + K(t)(y - C(t)\hat{x}).$$

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$$\dot{\hat{x}} = A\hat{x} + B_1 g + B_2 \dot{a} + K(t)(y - C(t)\hat{x}).$$

- $K(t) = \gamma L_\gamma P(t) C(t)^\top Q(t)$, with $\gamma \geq 1$, $L_\gamma = \mathrm{blockdiag}(I_3, \gamma I_3, \gamma^2 I_3)$ and $P(t)$ is solution of the CRE

$$\frac{1}{\gamma}\dot{P} = AP + PA^\top - PC(t)^\top Q(t) C(t) P + V(t),$$

Consider the "nonlinear complementary filter"-type observer

$$\dot{\hat{R}} = \hat{R}[\omega + \sigma_R]_\times$$

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$$\sigma_R = \rho_1(m_B \times \hat{R}^\top m_I) + \rho_2(a^{\mathcal{B}} \times \hat{R}^\top \mathbf{sat}(\hat{a})).$$

- **sat**: is a saturation function

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H. F. Grip, T. I. Fossen, T. A. Johansen and A. Saberi, "**Attitude Estimation Using Biased Gyro and Vector Measurements With Time-Varying Reference Vectors**," in IEEE TAC, 2012.

Universal Observers: Inertial-Frame Position Information

Stability Result # 2

Assumption

- *Pair $(A, C(\cdot))$ is uniformly observable*
- *m and $a(t)$ are not collinear for all times*
- *$\omega, a, \dot{a}$ are uniformly bounded*

Theorem (Almost semi-global exponential stability)

For all initial conditions (except attitude errors at 180°), there exist (high) gains such that the estimation error converges exponentially to zero.

S. Berkane, A. Tayebi, and S. De Marco, "A nonlinear navigation observer using imu and generic position information," *Automatica*, vol. 127, p. 109 513, 2021.

- Although, sufficiently large gains are needed in the proof, simulation results demonstrate that this is very conservative.

Lemma (Uniform observability)

$(A, C(t))$ is uniformly observable if there exist $\delta, \mu > 0$ such that for all $t \geq 0$ one has

$$\frac{1}{\delta}\int_{t}^{t+\delta}\sum_{i=1}^{n}\Pi(R_i b_i(s))ds \geq \mu I_3.$$

Hamel, Tarek, and Claude Samson. "Position estimation from direction or range measurements." Automatica 82 (2017): 137-144.

- **Single camera:** Vehicle is never static nor moving on a straight line passing through the camera.
- **Two cameras:** Vehicle not aligned (indefinitely) with both cameras.
- **Three cameras or more:** If at least 3 cameras are not aligned, the observability condition is always satisfied.

Universal Observers: Inertial-Frame Position Information

UO for bearing measurements

Lemma (Uniform observability)

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$$\frac{1}{\delta}\int_{t}^{t+\delta}\sum_{i=1}^{n}\Pi(R_i b_i(s))ds \geq \mu I_3.$$

Hamel, Tarek, and Claude Samson. "Position estimation from direction or range measurements." Automatica 82 (2017): 137-144.

- **Single camera:** Vehicle is never static nor moving on a straight line passing through the camera.
- **Two cameras:** Vehicle not aligned (indefinitely) with both cameras.
- **Three cameras or more:** If at least 3 cameras are not aligned, the observability condition is always satisfied.

UO for bearing measurements+ Altimeter

- An altimeter is a sensor that measures the altitude

$$y = \begin{bmatrix} C_p(t) \\ e_3^\top \end{bmatrix} p.$$

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$$y = \begin{bmatrix} C_p(t) \\ e_3^\top \end{bmatrix} p.$$

Lemma (Uniform observability: bearings+altimeter)

$(A, C(t))$ *is uniformly observable if there exist* $\delta, \mu > 0$ *such that for all* $t \geq 0$ *one has*

$$\frac{1}{\delta}\int_t^{t+\delta}\sum_{i=1}^{n}\Pi(R_i b_i(s))ds + e_3 e_3^\top \geq \mu I_3.$$

Universal Observers: Inertial-Frame Position Information

UO for bearing measurements+ Altimeter

- An altimeter is a sensor that measures the altitude

$$y = \begin{bmatrix} C_p(t) \\ e_3^\top \end{bmatrix} p.$$

Lemma (Uniform observability: bearings+altimeter)

$(A, C(t))$ *is uniformly observable if there exist* $\delta, \mu > 0$ *such that for all* $t \geq 0$ *one has*

$$\frac{1}{\delta}\int_t^{t+\delta} \sum_{i=1}^{n} \Pi(R_i b_i(s))ds + e_3 e_3^\top \geq \mu I_3.$$

- **Single camera + altimeter**: Vehicle not moving at the same altitude as the camera or, if doing so, not moving on a straight line passing through the camera
- **Two cameras + altimeter**: Cameras not at the same altitude. If the cameras are at the same altitude, vehicle must not be aligned (indefinitely) with both cameras.

Universal Observers: Inertial-Frame Position Information

Simulations: single bearing

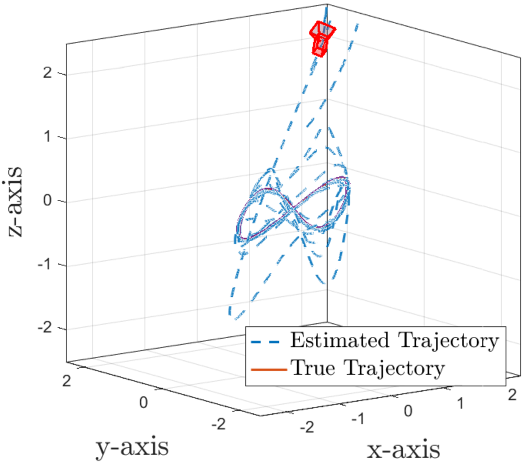


Figure 4: Observability holds but slow convergence.

Simulations: single bearing + altimeter

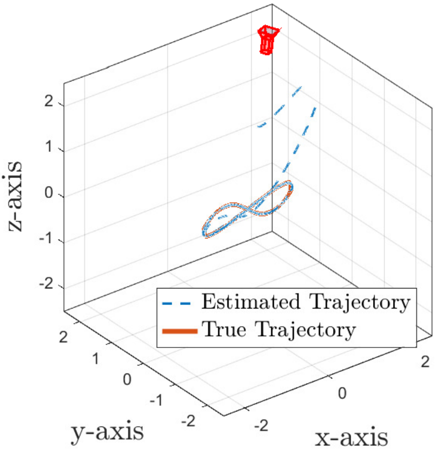
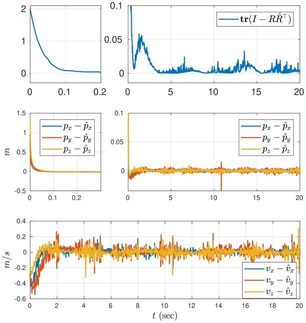
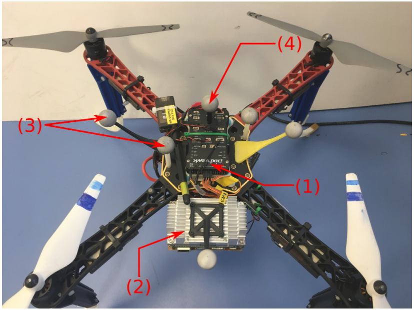


Figure 5: Stronger observability.

Experimental results





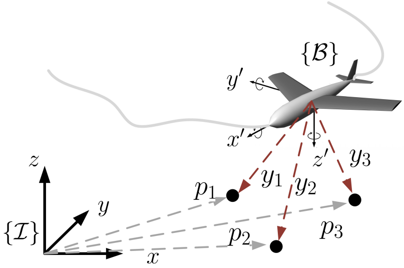
at University of Nice Sophia Antipolis

S. Berkane, A. Tayebi, and S. De Marco, "A nonlinear navigation observer using imu and generic position information," *Automatica*, vol. 127, p. 109 513, 2021.

Outline

Inertial Navigation Systems

Vision-Aided Inertial Navigation Systems (VINS)



Vision-Aided Inertial Navigation

- Useful in GPS-denied environments
- Information on the surrounding environment

- Body-frame landmark position measurements:

$$y_i = R^\top (p_i - p), \quad i = 1, \ldots, N$$

with at least $N \geq 3$ non-collinear landmarks

Inertial Navigation Systems

Vision-Aided Inertial Navigation Systems (VINS): Stereo vs. Monocular

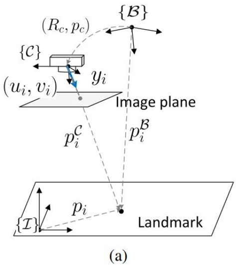


Figure 6: (a) monocular vision system (b) stereo vision system.

remark

Landmark positions can be algebraically reconstructed from stereo-bearing measurements. The direct use of the stereo-bearing measurements gives **robustness** *in situations where one of the cameras of the stereo-vision system loses sight of the landmarks for some period of time.*

Let p_i denote the (constant and known) position of the i-th landmark in frame $\{\mathcal{I}\}$, and

$$p_i^{\mathcal{B}} := R^\top (p_i - p)$$

denote the position of the i-th landmark in frame $\{\mathcal{B}\}$

1) **Stereo-bearing measurements:**

$$y_i^s := \frac{p_i^{\mathcal{C}_s}}{\|p_i^{\mathcal{C}_s}\|} = \frac{R_{cs}^\top (p_i^{\mathcal{B}} - p_{cs})}{\|p_i^{\mathcal{B}} - p_{cs}\|}, \; i \in \{1, 2, \dots, N\}, s \in \{1, 2\} \tag{21}$$

where $p_i^{\mathcal{C}_s} = R_{cs}^\top (p_i^{\mathcal{B}} - p_{cs})$.

2) **Monocular-bearing measurements:**

$$y_i := \frac{p_i^{\mathcal{C}}}{\|p_i^{\mathcal{C}}\|} = \frac{R_c^\top (p_i^{\mathcal{B}} - p_c)}{\|p_i^{\mathcal{B}} - p_c\|}, \; i \in \{1, 2, \dots, N\} \tag{22}$$

where $p_i^{\mathcal{C}} = R_c^\top (p_i^{\mathcal{B}} - p_c)$.

Universal Observers: Body-Frame Position Information

Inertial-Frame Kinematics

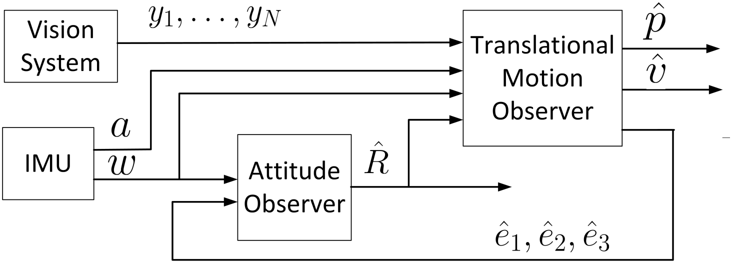


Figure 7: Structure of the proposed nonlinear observer on $SO(3) \times \mathbb{R}^{15}$ for vision-aided INS.

remark

The **auxiliary signals** $\hat{e}_1, \hat{e}_2, \hat{e}_3$ *are introduced to help* **construct the attitude innovation term** *due to the lack of known inertial-frame vectors (in the case of bearing measurements).*

$$\dot{\hat{R}} = \hat{R}(\omega + \hat{R}^\top \sigma_R)^\times \tag{23a}$$

$$\dot{\hat{p}} = \hat{v} + \sigma_R^\times \hat{p} + \hat{R} K_p \sigma_y \tag{23b}$$

$$\dot{\hat{v}} = \hat{g} + \hat{R} a + \sigma_R^\times \hat{v} + \hat{R} K_v \sigma_y \tag{23c}$$

$$\dot{\hat{e}}_i = \sigma_R^\times \hat{e}_i + \hat{R} K_i \sigma_y, \quad i = 1, 2, 3 \tag{23d}$$

The attitude innovation term σ_R is given as follows:

$$\sigma_R := \frac{k_R}{2} \hat{R} \sum_{i=1}^{3} \rho_i (\hat{R}^\top \hat{e}_i)^\times (\hat{R}^\top e_i) = \frac{k_R}{2} \sum_{i=1}^{3} \rho_i \hat{e}_i^\times e_i \tag{24}$$

The gain matrices $K_p, K_v, K_i \in \mathbb{R}^{3\times 3}, i = 1, 2, 3$ are designed as follows:

$$K = PC^\top(t) Q(t) = [K_p^\top, K_1^\top, K_2^\top, K_3^\top, K_v^\top]^\top \tag{25}$$

The matrix P is the solution to the following CRE:

$$\dot{P} = A(t)P + PA^\top(t) - PC^\top(t)Q(t)C(t)P + V(t) \tag{26}$$

$$A(t) = \begin{bmatrix} -\omega^\times & 0_3 & 0_3 & 0_3 & I_3 \\ 0_3 & -\omega^\times & 0_3 & 0_3 & 0_3 \\ 0_3 & 0_3 & -\omega^\times & 0_3 & 0_3 \\ 0_3 & 0_3 & 0_3 & -\omega^\times & 0_3 \\ 0_3 & g_1 I_3 & g_2 I_3 & g_3 I_3 & -\omega^\times \end{bmatrix}. \tag{27}$$

From the monocular-bearing measurements, vector σ_{yi} is designed as

$$\sigma_{yi} = \Pi(R_c y_i)(\hat{R}^\top(\hat{p}_i - \hat{p}) - p_c) \tag{28}$$

with $\Pi_i := \Pi(R_c y_i) \in \mathbb{R}^{3\times 3}, i \in \{1, \ldots, N\}$. The matrix $C(t)$ is given by:

$$C(t) = \begin{bmatrix} \Pi_1 & -p_{11}\Pi_1 & -p_{12}\Pi_1 & -p_{13}\Pi_1 & 0_3 \\ \Pi_2 & -p_{21}\Pi_2 & -p_{22}\Pi_2 & -p_{23}\Pi_2 & 0_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Pi_N & -p_{N1}\Pi_N & -p_{N2}\Pi_N & -p_{N3}\Pi_N & 0_3 \end{bmatrix} \tag{29}$$

- The proof of AGAS is based on the following geometric errors

$$\tilde{R} = R\hat{R}^\top \in SO(3)$$
$$\tilde{p} = R^\top p - \hat{R}^\top \hat{p}$$
$$\tilde{v} = R^\top v - \hat{R}^\top \hat{v}$$
$$\tilde{e}_i = R^\top e_i - \hat{R}^\top \hat{e}_i, \forall i \in \{1, 2, 3\}$$

with $\tilde{x} := [\tilde{p}^\top, \tilde{e}_1^\top, \tilde{e}_2^\top, \tilde{e}_3^\top, \tilde{v}^\top]^\top \in \mathbb{R}^{15}$

remark

The projection operator $\Pi(R_c y_i)$ allows to eliminate the component which is collinear to $R_c y_i$. The projection of $\hat{R}^\top(\hat{p}_i - \hat{p}) - p_c$ onto the plane orthogonal to $R_c y_i$ will boil down to the projection of $\hat{R}^\top(\hat{p}_i - \hat{p}) - p_i^{\mathcal{B}}$. This mechanism, allow us to **put the error injection vector as** *$\sigma_y = C(t)\tilde{x}$.*

- One obtains the following closed-loop system:

$$\dot{\tilde{R}} = \tilde{R}(-\psi_a(M\tilde{R}) - \Gamma(t)\tilde{x})^\times \quad (30a)$$

$$\dot{\tilde{x}} = (A(t) - KC(t))\tilde{x} \quad (30b)$$

Theorem

Assume $(A(t), C(t))$ **is uniformly observable**. *Then:*

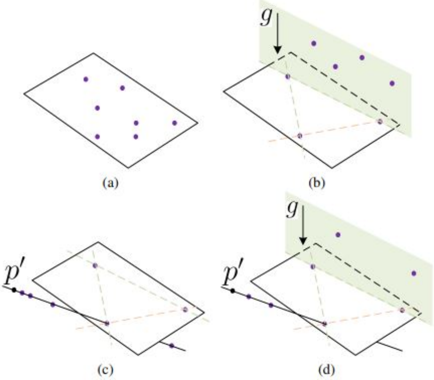
i) *All solutions of the closed-loop system* (30) *converge to the set of equilibria given by* $(I_3, 0_{15\times 1}) \cup \Psi_M$ *where*

$$\Psi_M := \{(\tilde{R}, \tilde{x}) \in SO(3) \times \mathbb{R}^{15} | \tilde{R} = \mathcal{R}_\alpha(\pi, v), v \in \mathcal{E}(M), \tilde{x} = 0_{15\times 1}\}.$$

ii) *The desired equilibrium* $(I_3, 0_{15\times 1})$ *is locally exponentially stable.*

iii) *All the undesired equilibria in* Ψ_M *are unstable, and the desired equilibrium* $(I_3, 0_{15\times 1})$ *is almost globally asymptotically stable.*

Motionless monocular camera



Monocular camera in motion

- there exist three non-aligned landmarks whose plane is not parallel to the gravity vector
- the camera is regularly not aligned with the same landmark

M. Wang, S. Berkane, and A. Tayebi, "Nonlinear observers design for vision-aided inertial navigation systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 4, pp. 1853–1868, 2021.

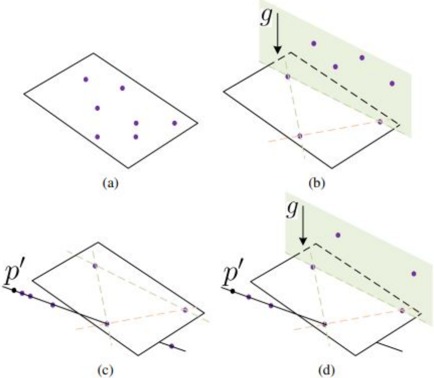
Algorithm 1 Nonlinear observer for vision-aided INSs

Input: Continuous IMU measurements, and intermittent visual measurements at the time instants $\{t_k\}_{k\in\mathbb{N}_{>0}}$.

Output: $\hat{R}(t), \hat{p}(t)$ and $\hat{v}(t)$ for all $t \geq 0$

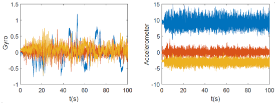
```
for k ≥ 1 do
    while t ∈ [t_{k-1}, t_k] do
        dR̂ = R̂(ω + R̂^⊤ σ_R)^×                 /* σ_R defined in (7) */
        dp̂ = σ_R^× p̂ + v̂
        dv̂ = σ_R^× v̂ + Σ_{i=1}^{3} g_i ê_i + R̂a
        dê_i = σ_R^× ê_i                        /* for all i = 1,2,3 */
        dP = A(t)P + PA^⊤(t) + V(t)             /* A defined in (10)
            and V(t) being uniformly positive definite */
    end while
    Obtain the vector σ_y and matrix C(t) from the visual
    measurements at time t_k                   /*Using (11) and
    (12) for stereo-bearing measurements, or (13) and (14)
    for monocular-bearing measurements */
    K = PC^⊤(t_k)(C(t_k)PC^⊤(t_k) + Q^{-1}(t_k))^{-1}   /* Q(t)
    being uniformly positive definite */
    Compute matrices K_p, K_1, K_2, K_3 and K_v from K
    /*using K = [K_p^⊤, K_1^⊤, K_2^⊤, K_3^⊤, K_v^⊤]^⊤ */
    R̂^+ = R̂
    p̂^+ = p̂ + R̂K_p σ_y
    v̂^+ = v̂ + R̂K_v σ_y
    ê_i^+ = ê_i + R̂K_i σ_y                      /* for all i = 1,2,3 */
    P^+ = (I_15 − KC)P
end for
```

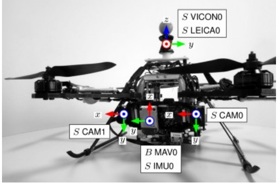
Motionless monocular camera

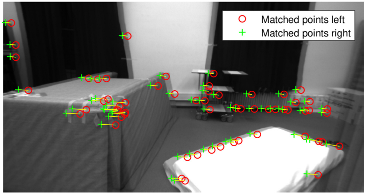


EuRoc dataset:

- The trajectories are generated from a real quadrotor flight
- The EuRoc dataset includes a set of stereo images (20Hz) and IMU measurements (200Hz)
- The ground truth is obtained from Vicon motion capture system

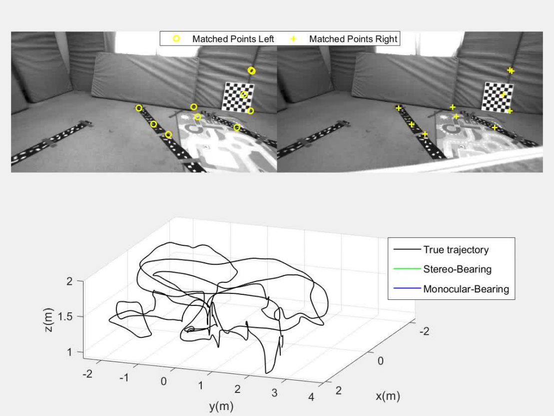






Universal Observers: Body-Frame Position Information

Inertial-Frame Kinematics: Experimental Results



- **Stereo-Bearings** position error $\approx 3.26cm$
- **Monocular-Bearings** position error $\approx 10.99cm$

M. Wang, S. Berkane, and A. Tayebi, "Nonlinear observers design for vision-aided inertial navigation systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 4, pp. 1853–1868, 2021.

- Model with body-frame states

$$
\begin{aligned}
\dot{p}^{\mathcal{B}} &= -[\omega]_\times p^{\mathcal{B}} + v^{\mathcal{B}}, && (31)\\
\dot{v}^{\mathcal{B}} &= -[\omega]_\times v^{\mathcal{B}} + a^{\mathcal{B}} + \sum g_i e_i^{\mathcal{B}}, && (32)\\
\dot{e}_1^{\mathcal{B}} &= -[\omega]_\times e_1^{\mathcal{B}}, && (33)\\
\dot{e}_2^{\mathcal{B}} &= -[\omega]_\times e_2^{\mathcal{B}}, && (34)\\
\dot{e}_3^{\mathcal{B}} &= -[\omega]_\times e_3^{\mathcal{B}}. && (35)
\end{aligned}
$$

- Let us now introduce the following virtual output:

$$
y := \begin{bmatrix} \Pi_1(p^{\mathcal{B}} - p_1^{\mathcal{B}}) \\ \vdots \\ \Pi_N(p^{\mathcal{B}} - p_N^{\mathcal{B}}) \end{bmatrix} = \begin{bmatrix} 0_{3\times 1} \\ \vdots \\ 0_{3\times 1} \end{bmatrix} \qquad (36)
$$

- Define the following state vector

$$x = \left[(p^{\mathcal{B}})^\top \quad (e_1^{\mathcal{B}})^\top \quad (e_2^{\mathcal{B}})^\top \quad (e_3^{\mathcal{B}})^\top \quad (v^{\mathcal{B}})^\top\right]^\top, \tag{37}$$

- We obtain an LTV system of the form:

$$\dot{x} = A(t)x + Ba^{\mathcal{B}}, \tag{38a}$$

$$y = C(t)x, \tag{38b}$$

with matrices $A(t), B$ and $C(t)$ defined previously.

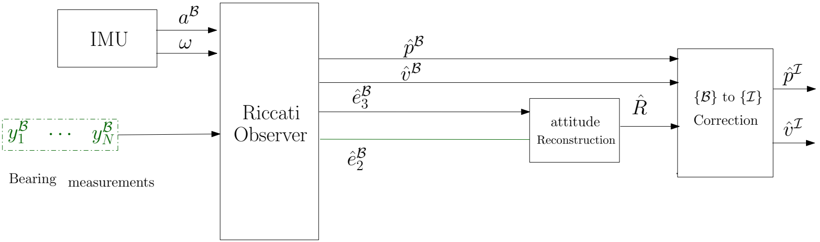
- The state of system is then estimated using the Riccati observer:

$$\dot{\hat{x}} = A(t)\hat{x} + Ba^{\mathcal{B}} + K(t)(y - C(t)\hat{x}), \tag{39}$$

Theorem

If $(A(t), C(t))$ is UO then we have UGES.

Body-Frame Kinematics



- The orientation matrix can be computed using algebraic reconstruction

$$\hat{R}^\top = \left[\hat{e}_2^{\mathcal{B}} \times \hat{e}_3^{\mathcal{B}} \quad \hat{e}_3^{\mathcal{B}} \times (\hat{e}_2^{\mathcal{B}} \times \hat{e}_3^{\mathcal{B}}) \quad \hat{e}_3^{\mathcal{B}}\right], \tag{40}$$

- $\hat{R}$ is then projected on $SO(3)$ using polar decomposition

P. Martin and I. Sarras, "A global observer for attitude and gyro biases from vector measurements," *20th IFAC World Congress,*, 2017.

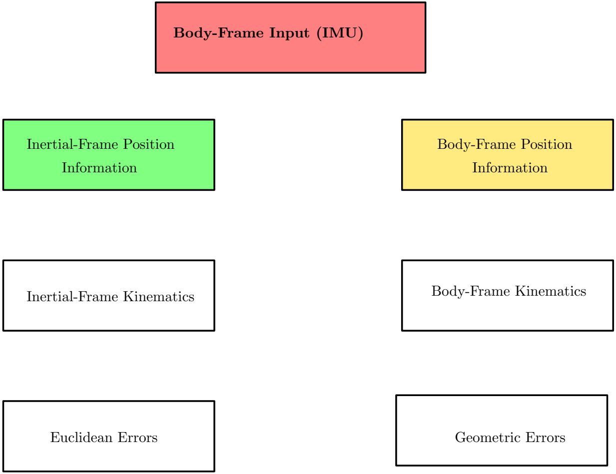
- This approach allows the use of any attitude reconstruction/estimation algorithm.
- For example, one can use a complementary filter on $SO(3)$ to recover the attitude with AGAS result:

$$\begin{cases} \dot{\hat{R}} & = \hat{R}[\omega + \sigma_R]_\times, \\ \sigma_R & = \sum \rho_i(\hat{e}_i^{\mathcal{B}} \times \hat{R}^\top e_i) \end{cases}$$

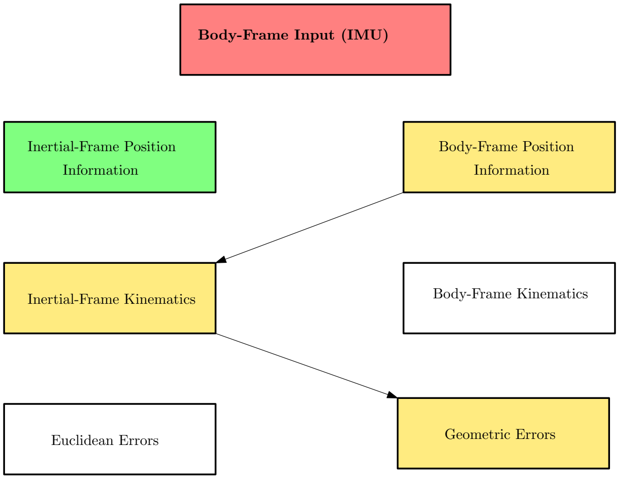
- This approach can be tailored to double or even single bearing measurements

S.-E. Benahmed and S. Berkane, "State estimation using single body-frame bearing measurements," in *European Control Conference*, 2024.

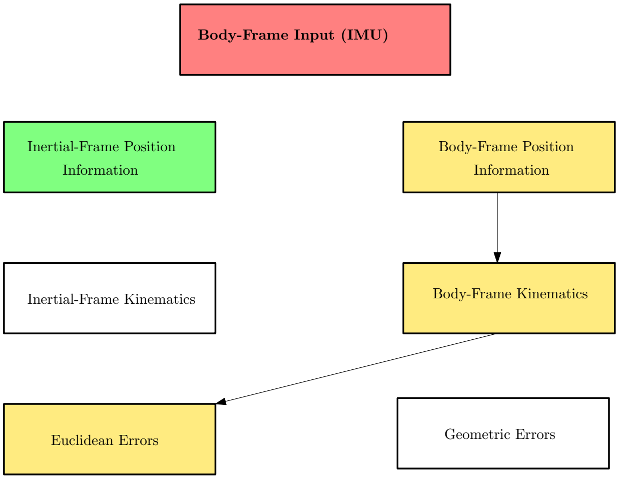
Concluding Remarks



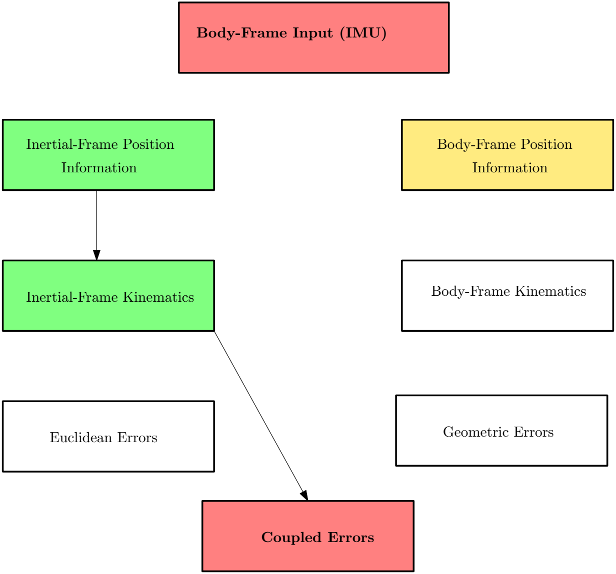
Concluding Remarks



Concluding Remarks



Concluding Remarks



Thank you

Questions?