# Navigation in Unknown Environments Using Safety Velocity Cones

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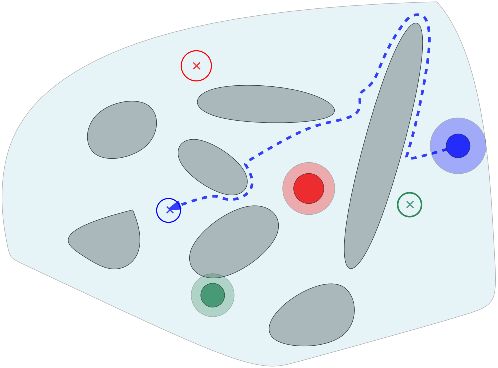
# Safe Navigation in Unknown Environement



- Cluttered and unknown (2D, 3D) environment
- Local sensor information (*e.g., relative bearing, range*)
- More generic obstacle shapes (*e.g. non-convex, non-smooth*)
- On-board computation

# Outline

Nagumo's Theorem for Invariant Sets

Projection onto tangent cones for safe navigation

Distance-based characterization and continuous vector fields generation

Sensor-based implementation

# Nagumo's theorem for invariant sets

Consider the system

$$\dot{x} = f(x), \quad x(0) \in \mathcal{X} \subset \mathbb{R}^n \tag{1}$$

**Theorem (Nagumo, 1942, sub-tangentiality condition)**

*Assume solutions are* **unique** *and the set* $\mathcal{X}$ *is* **closed**. *Then, the set* $\mathcal{X}$ *is forward invariant if and only if*

$$f(x) \in \mathbf{T}_{\mathcal{X}}(x), \quad \forall x \in \mathcal{X} \tag{2}$$

- Bouligand's tangent cone
  $\mathbf{T}_{\mathcal{X}}(x) := \left\{ z : \lim_{\tau \to 0^+} \inf \frac{\mathbf{d}_{\mathcal{S}}(x+\tau z)}{\tau} = 0 \right\}$

# Nagumo's theorem for invariant sets

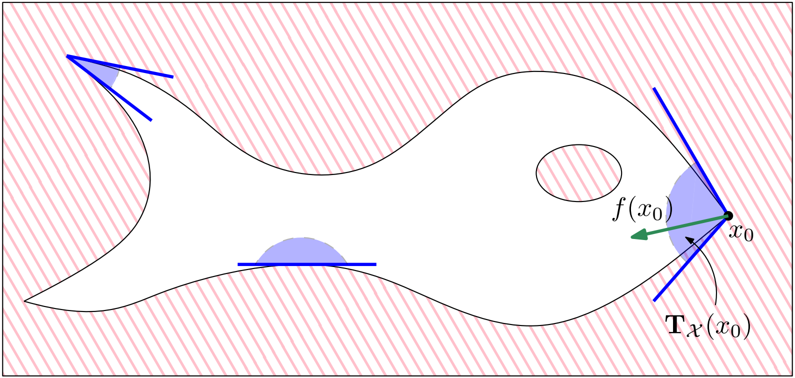



Figure 1: Nagumo's condition applied to a fish shaped set.

# Projection onto tangent cones for safe navigation

## Problem formulation

Consider the kinematic system

$$\dot{x} = u, \quad u \in \mathbb{R}^n.$$

- A closed $\mathcal{X} \subset \mathbb{R}^n$: free workspace
- **Objectives:** design $u = \kappa_{\mathcal{X}}(x)$ such that:
  1. $x \to x_d$ (asymptotically)
  2. $x$ stays in $\mathcal{X}$ (safety)
  3. the control policy $\kappa_{\mathcal{X}}$ must be locally computed
- By Nagumo's theorem and to ensure safety, we need

$$u \in \mathbf{T}_{\mathcal{X}}(x), \quad \forall x \in \mathcal{X}.$$

- To ensure convergence to the target, the nominal controller is

$$\kappa_0(x) = -k(x - x_d), \quad k > 0.$$

# Projection onto tangent cones for safe navigation

## Invariance through safety velocity cones (SVCs)

Therefore, it is reasonable to consider the following optimization

$$\min_{u} \|u - \kappa_0(x)\|^2 \quad \text{subject to } u \in \mathbf{T}_{\mathcal{X}}(x), \forall x \in \mathcal{X}$$

- $u$ is the **projection** of $\kappa_0(x)$ onto the tangent cone $\mathbf{T}_{\mathcal{X}}(x)$
- $\mathbf{T}_{\mathcal{X}}(x)$ is referred to as **safety velocity cone (SVC)**.
- Since $\mathbf{P}_{\mathbf{T}_{\mathcal{X}}(x)}(\cdot)$ is set-valued, the resulting closed-loop system can be written as a differential inclusion

$$\dot{x} = \kappa_{\mathcal{X}}(x) \in \mathbf{P}_{\mathbf{T}_{\mathcal{X}}(x)}(\kappa_0(x)), \tag{3}$$

$$= \begin{cases} \{\kappa_0(x)\}, & x \in \mathbf{int}(\mathcal{X}) \\ \mathbf{P}_{\mathbf{T}_{\mathcal{X}}(x)}(\kappa_0(x)), & x \in \partial\mathcal{X}, \end{cases} \tag{4}$$

# Projection onto tangent cones for safe navigation

## Invariance through safety velocity cones (SVCs)

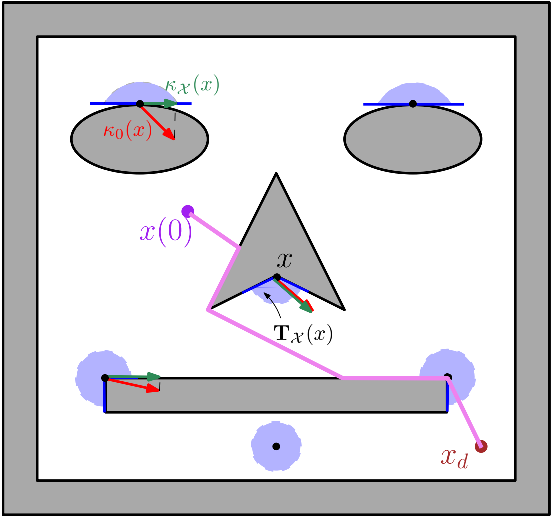



Figure 2: Set with convex tangent cones at each $x$. The nominal controller is projected onto the tangent cone.

# Projection onto tangent cones for safe navigation

## Invariance through safety velocity cones (SVCs)

- In view of the famous Hilbert project theorem, if the tangent cone $\mathbf{T}_{\mathcal{X}}(x)$ is convex at each $x$, the projection map $\mathbf{P}_{\mathbf{T}_{\mathcal{X}}(x)}(\cdot)$ is **single-valued**.
- Arbitrary closed sets with smooth boundaries have tangent cones that are half-spaces (thus convex)
- In the 2D context, the obtained navigation trajectories are similar to those obtained using the traditional Bug planning algorithm where the planner switches between motion-to-goal and boundary-following maneuvers
- Our proposed obstacle avoidance strategy extends this to an Euclidean space with arbitrary dimension while also uniting the planning and feedback stabilization tasks

# Projection onto tangent cones for safe navigation

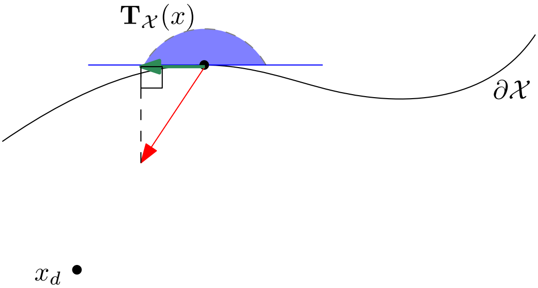
## Free spaces with smooth boundaries

- For smooth boundaries, there exists a continuously differentiable map (Gauss map) $\nu : \partial\mathcal{X} \to \mathbb{S}^{n-1}$

$$\mathbf{T}_{\mathcal{X}}(x) = \left\{ z : z^\top \nu(x) \leq 0 \right\} \quad \text{(half-space)} \tag{5}$$

- The projection (on the boundary) is given by

$$\Pi(\nu(x))\kappa_0(x) := (I_n - \nu(x)\nu(x)^\top)\kappa_0(x), \quad \nu(x)^\top \kappa_0(x) \geq 0$$



# Projection onto tangent cones for safe navigation

## Free spaces with smooth boundaries

**Theorem**

*Consider a free space that is described by a closed set $\mathcal{X} \subset \mathbb{R}^n$ such that $\partial\mathcal{X}$ is an orientable and $C^2-$hypersurface.*

1. *Closed-loop system admits a unique solution (in the sense of Fillipov)*
2. *The set $\mathcal{X}$ is forward invariant.*
3. *The distance $\|x - x_d\|$ is non-increasing.*
4. *The equilibrium $x = x_d$ is (locally) exponentially stable.*
5. *Solution converges to the set $\{x_d\} \cup \mathcal{E}$, where $\mathcal{E} := \{x \in \partial\mathcal{X} : x = x_d + \lambda\nu(x), \lambda \in \mathbb{R}_{<0}\}$*

- The subset $\mathcal{E} \subset \partial\mathcal{X}$ is Lebesgue measure zero.
- Studying the invariance properties of the undesired equilibria depends on the considered set $\mathcal{X}$ (e.g., topological properties).

# Projection onto tangent cones for safe navigation

## Euclidean sphere worlds

**Theorem (Euclidean Sphere Worlds)**

*If the obstacles are spherical, we further have*

1. *The unique Filippov solution converges to the set* $\{x_d\} \cup_{i=1}^{M} \{\bar{x}_i\}$ *with* $\bar{x}_i := (1 - \alpha_i)x_d + \alpha_i c_i$ *and* $\alpha_i := 1 + r_i \|x_d - c_i\|^{-1}$.
2. *Each undesired equilibrium point* $x = \bar{x}_i$ *is unstable.*
3. *The desired equilibrium* $x = x_d$ *is locally exponentially stable and almost globally asymptotically stable.*

- Geometrically speaking, the unstable equilibrium $\bar{x}_i$ is nothing but the antipodal point (on the boundary of the obstacle) which is diametrically opposite to $x_d$

# Distance-based characterization and continuous vector fields generation

- We pick a safety margin $\epsilon > 0$ and define

$$\mathcal{X}_\epsilon := \{x \in \mathbb{R}^n : \mathbf{d}_{\complement\mathcal{X}}(x) \geq \epsilon\} \tag{6}$$

- The outward normal vector $\nu(x)$ at $\partial\mathcal{X}_\epsilon$ is nothing but

$$\nu(x) = -\nabla \mathbf{d}_{\complement\mathcal{X}}(x) = \frac{\mathbf{P}_{\partial\mathcal{X}}(x) - x}{\mathbf{d}_{\complement\mathcal{X}}(x)}. \tag{7}$$

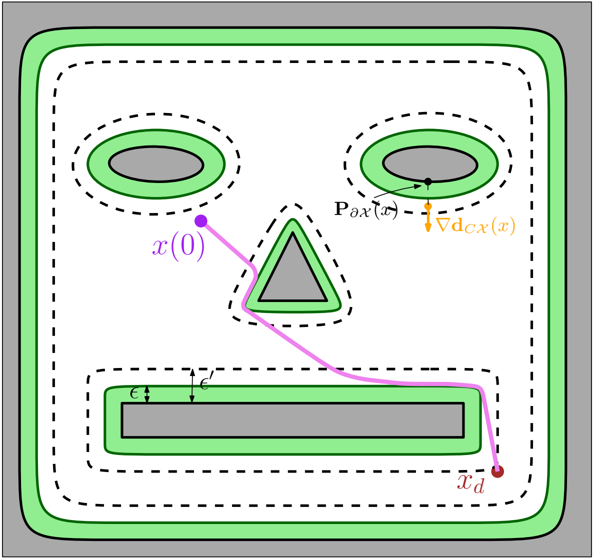
- The controller can be implemented using two local information: projection position onto the boundary $\mathbf{P}_{\partial\mathcal{X}}(x)$ and distance to the boundary $\mathbf{d}_{\complement\mathcal{X}}(x)$.
- The vector field can be made continuous by considering

$$\hat{\Pi}(x) := I_n - \phi(x)\nu(x)\nu(x)^\top, \tag{8}$$

$$\phi(x) := \min\left(1, \frac{\epsilon' - \mathbf{d}_{\complement\mathcal{X}}(x)}{\epsilon' - \epsilon}\right). \tag{9}$$

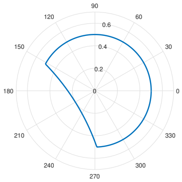
avec $0 < \epsilon < \epsilon'$.

# Distance-based characterization and continuous vector fields generation



# Sensor-based implementation

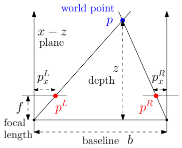
2D navigation using a LiDAR




$$\mathbf{d}_{\mathsf{C}\mathcal{X}}(x) = \min_{\theta} \rho(\theta; x)$$

$$\nabla \mathbf{d}_{\mathsf{C}\mathcal{X}}(x) = -(\cos(\theta^*), \sin(\theta^*))$$

3D navigation using stereo cam.




$$\text{depth} = \frac{f \cdot b}{|p_x^L - p_x^R|}$$

$$p_{x(y)} = \frac{b \cdot \bar{p}^L_{x(y)}}{(\bar{p}_x^L - \bar{p}_x^R)}$$

$$p_z = \frac{b \cdot f}{(\bar{p}_x^L - \bar{p}_x^R)}.$$

# Simulation example

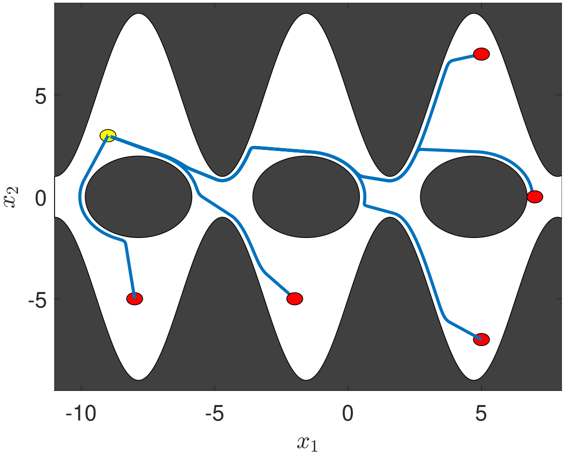



Figure 3: The obtained navigation trajectories of the proposed sensor-based control law, starting at a set of initial positions (red), converge to the goal (yellow) while avoiding the obstacles region. https://youtu.be/37ImfGoPyqg.

# Thank you

**Questions?**